Geodesy 1 (GED203)

Lecture No: 4

SOLUTION OF SPHERICAL TRIANGLES

Reda FEKRY

reda.abdelkawy@feng.bu.edu.eg

Review of Previous Lecture

COMMON CURVES ON SURFACE ELLIPSOID

NORMAL SECTION

GEODESIC LINE CURVE

RADII OF CURVATURE

LENGTHS OF DIFFERENT TYPES OF ARCS

SUMMARY

Where are we till now?

please keep up

OVERVIEW OF TODAY'S LECTURE

REVIEW OF PLANE TRIANGLE FORMULAS

PLANE TRIANGLES VERSUS SPHERICAL/GEODETIC TRIANGLES

SPHERICAL TRIANGLE

SOLUTION OF SPHERICAL TRIANGLE

NAPIER'S RULE FOR RIGHT-ANGLE SPHERICAL TRIANGLES

SPHERICAL EXCESS

CALCULATION OF SPHERICAL EXCESS

ELLIPSOIDAL EXCESS

SUMMARY

Expected Learning Outcomes

- Understanding the difference between plane triangles and spherical/geodetic triangles.
- Appreciating the need for specialized techniques to handle spherical or geodetic triangles.
- Identifying the unique properties and characteristics of spherical triangles.
- Learning various methods for solving spherical triangles, such as the Law of Sines, and Law of Cosines.
- Understanding and applying Napier's Rule to find the unknown angles and sides in right-angle spherical triangles.
- Recognizing that spherical excess is a measure of the deviation from planar geometry on a curved surface.
- Understanding the significance of spherical excess in geodesy and cartography.

What Do You Expect?

Plane Triangle

- **6 Elements**
  - Three Angles (α, β, γ): measured in angular units.
  - Three Sides (a, b, c): straight lines measured in linear units.
- **Sine Rule**

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

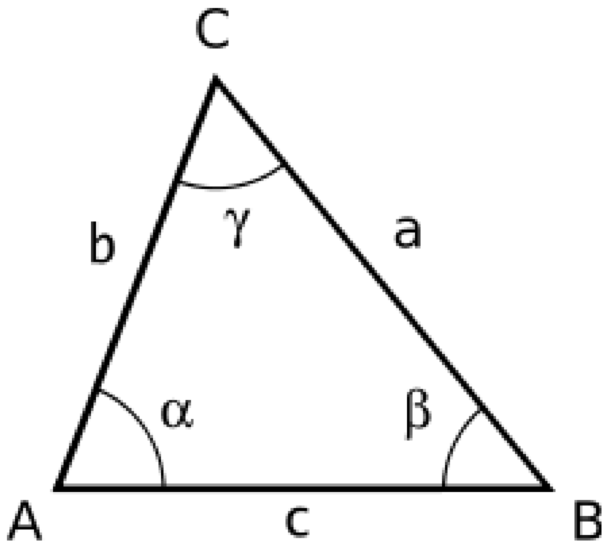
- **Cosine Rule**

$$a^2 = b^2 + c^2 - 2b \times c \times \cos\alpha$$

- **Area**

$$Area = \frac{1}{2} \times b \times c \times \sin\alpha = \frac{1}{2} \times a \times c \times \sin\beta = \frac{1}{2} \times a \times b \times \sin\gamma = \sqrt{s \times (s-a) \times (s-b) \times (s-c)}$$

Such that: $s = \frac{a+b+c}{2}$



Spherical Triangle

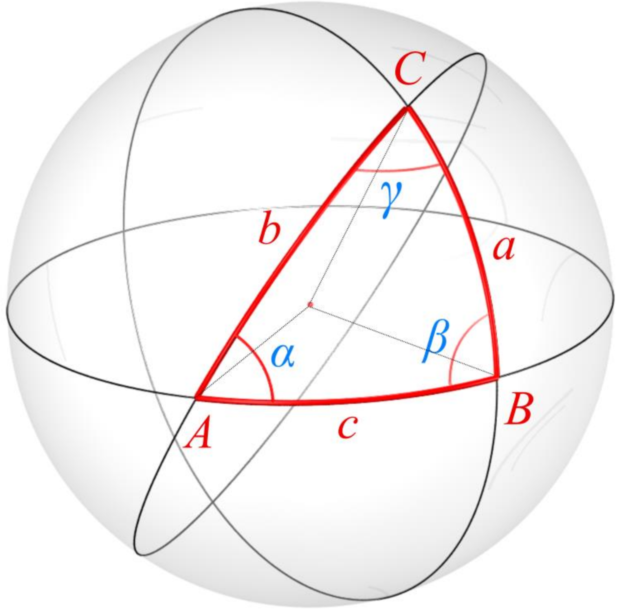
- **A spherical triangle** is a figure formed on the surface of a sphere by three great circular arcs intersecting pairwise in three vertices.
- **A spherical triangle has some elements: -**

a) *Three sides (a, b, and c):* measured in angular units.

b) *Three angles (A, B, and C):* measured in angular units.

c) *Three vertices (Intersection of sides)*

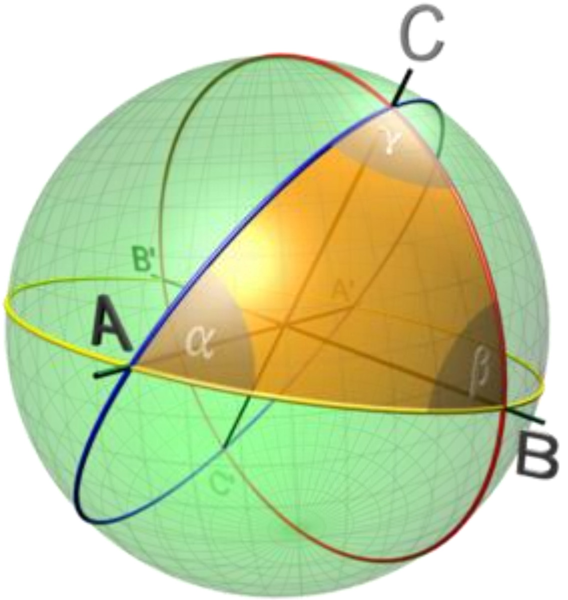
d) ***Spherical Excess*** ε (angle sums exceeding 180°)



Spherical Triangle

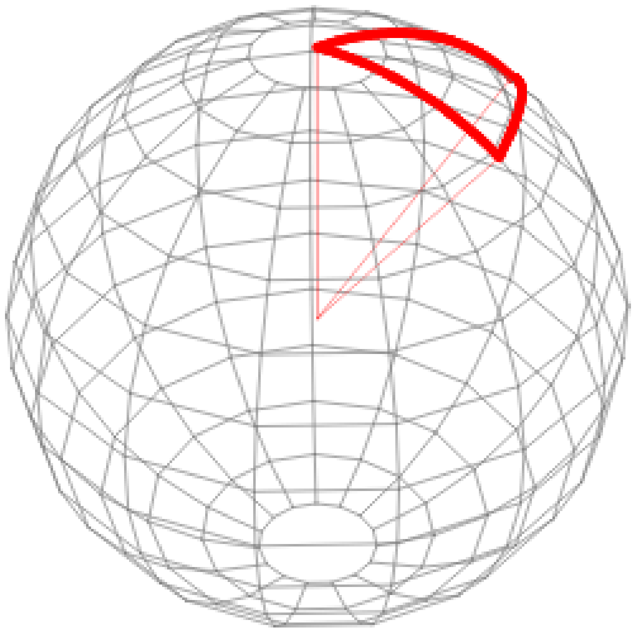
- **A triangle drawn on the surface of a sphere is only a spherical triangle if :**

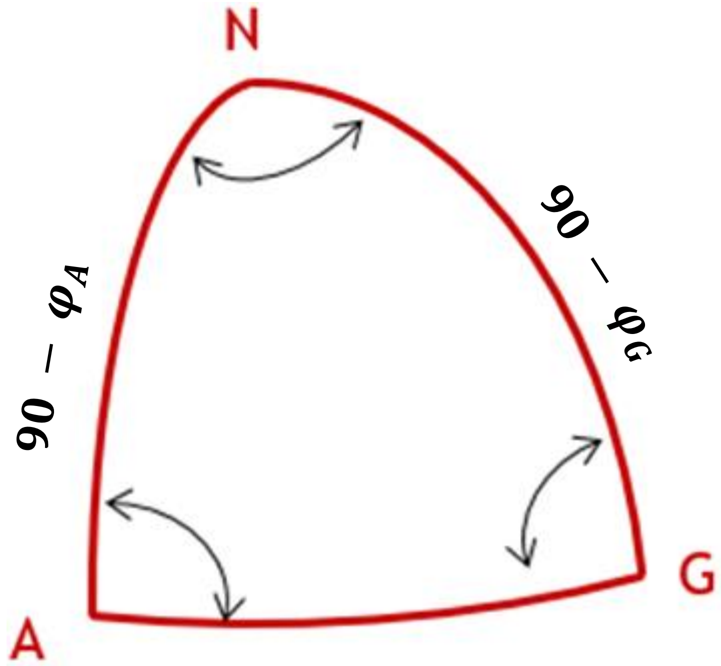
1) The three sides are all arcs of great circles.
2) Any two sides are together longer than the third side.
3) The sum of the three angles is greater than 180º.
4) Each individual spherical angle is less than 180º.



Spherical Triangle – Polar Triangle

- A polar triangle is a special case of a spherical triangle where one vertex is located at either the North Pole or the South Pole, and the sides are meridians of longitude.

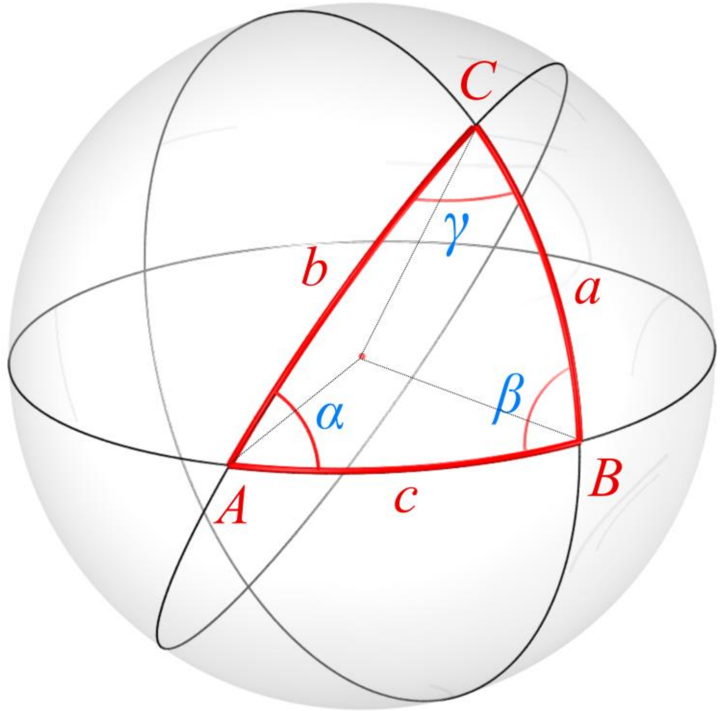




Solution of Spherical Triangle

- **Before applying law of sines or law of cosines, there are two types of angles in a spherical triangle: -**

1. Internal angles at vertices.
2. The sides, so we can take **sines** and **cosines** etc. of the **sides** as well as the vertex angles.



Solution of Spherical Triangle

- Law of sines:

$$\frac{\sin\alpha}{\sin a} = \frac{sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c}$$

- Law of cosines:

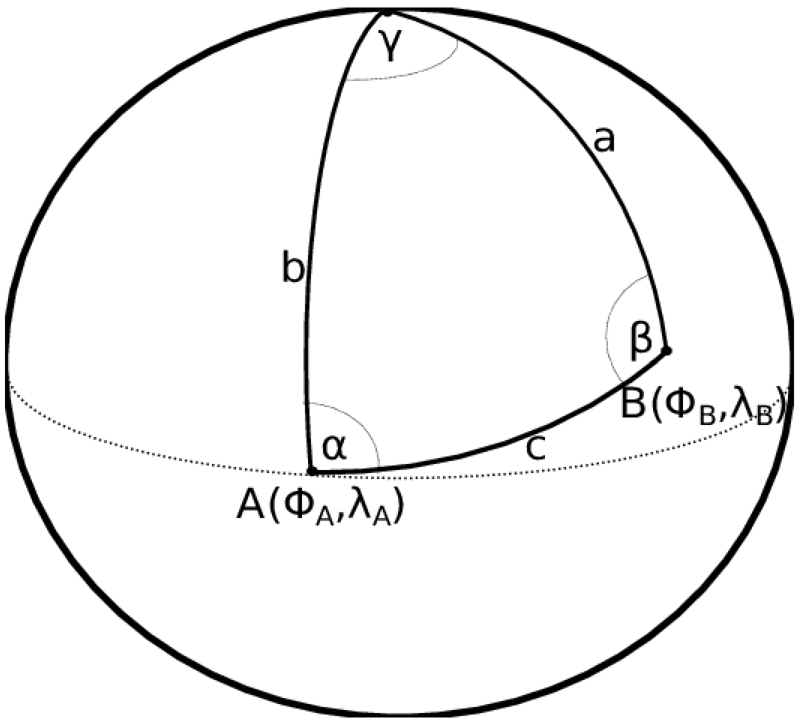
$$\cos a = \cos b \times \cos c + \sin b \times \sin c \times \cos\alpha$$

$$\cos\alpha = -\cos\beta \times \cos\gamma + \sin\beta \times \sin\gamma \times \cos a$$

- Tangents :

$$tan\,\frac{\alpha}{2} = \sqrt{\frac{sin(s-b)\times sin\,(s-c)}{sin\,s \times sin\,(s-a)}},\ s = \frac{a+b+c}{2}$$

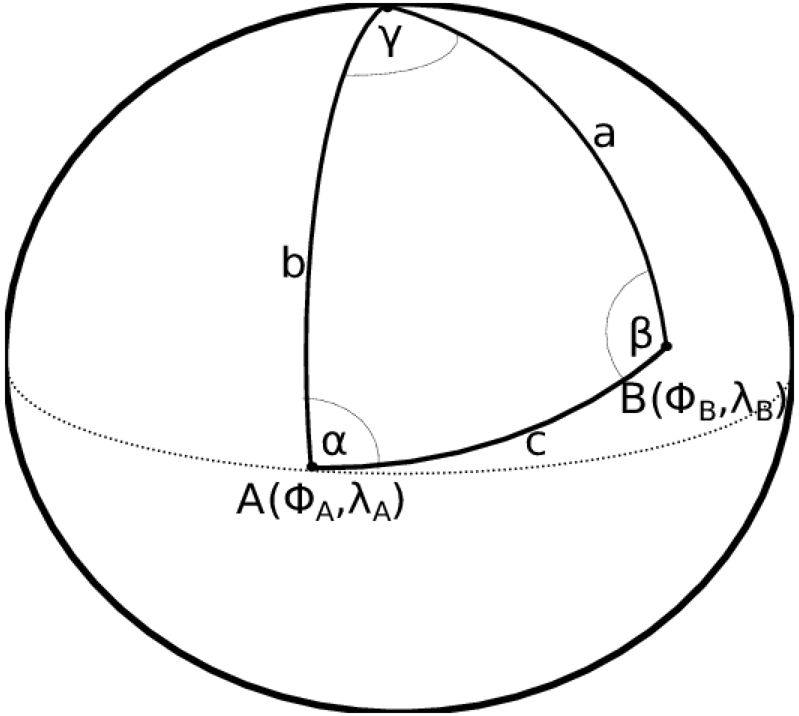
$$\tan\frac{a}{2} = \sqrt{\frac{-\cos\sigma \times \cos(\sigma-\alpha)}{\cos(\sigma-\beta) \times \cos(\sigma-\gamma)}},\ \sigma = \frac{\alpha+\beta+\gamma}{2}$$



Area of Spherical Triangle

$$Area = \frac{1}{2} \times a \times \ b \times \sin \gamma = \frac{1}{2} c^2 . \frac{\sin \alpha \times \sin \beta}{\sin(\alpha + \ \beta)}$$

$$= \sqrt{tan \frac{s}{2} \times \tan \frac{s-a}{2} \times \tan \frac{s \ - b}{2} \times \tan \frac{s-c}{2}}$$



[When to use each formula !!]

Common spherical trigonometry formulas

law of sines:	$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$
law of cosines:	$\cos a = \cos b \cos c + \sin b \sin c \cos A$ $\cos b = \cos a \cos c + \sin a \sin c \cos B$ $\cos c = \cos a \cos b + \sin a \sin b \cos C$
half-angle formulas:	$\tan\left(\frac{A}{2}\right) = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}$ $\tan\left(\frac{B}{2}\right) = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s \sin(s-b)}}$ $\tan\left(\frac{C}{2}\right) = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}}$, where $s = \frac{a+b+c}{2}$
half-side formulas:	$\tan\left(\frac{a}{2}\right) = \sqrt{\frac{-\cos S \cos(S-A)}{\cos(S-B)\cos(S-C)}}$ $\tan\left(\frac{b}{2}\right) = \sqrt{\frac{-\cos S \cos(S-B)}{\cos(S-A)\cos(S-C)}}$ $\tan\left(\frac{c}{2}\right) = \sqrt{\frac{-\cos S \cos(S-C)}{\cos(S-A)\cos(S-B)}}$, where $S = \frac{A+B+C}{2}$

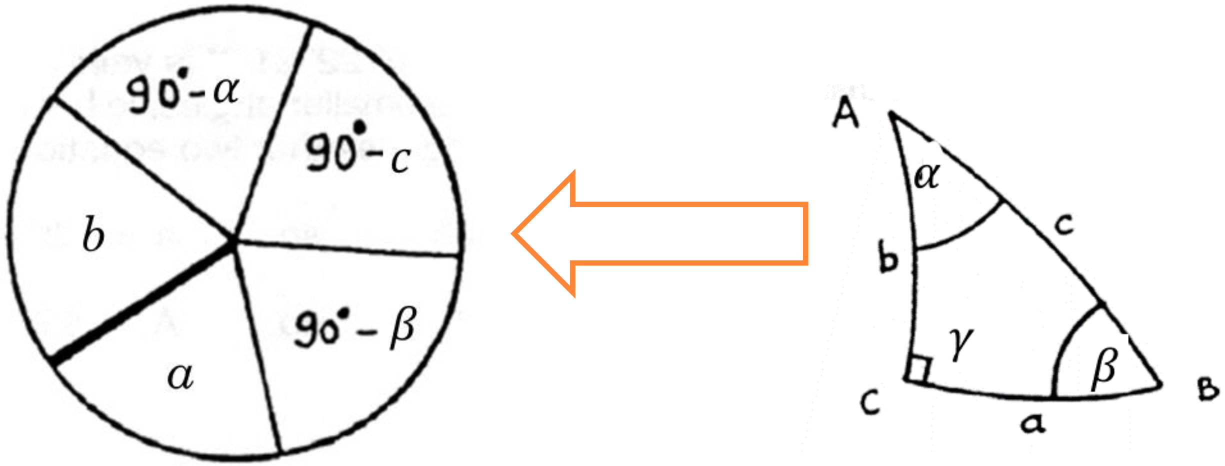
SOLUTION OF RIGHT-ANGLE SPHERICAL TRIANGLE

- **Napier's Rule**

Sine (Middle Part)

Product of Tangents of Adjacent Parts

Product of Cosines of Opposite Parts

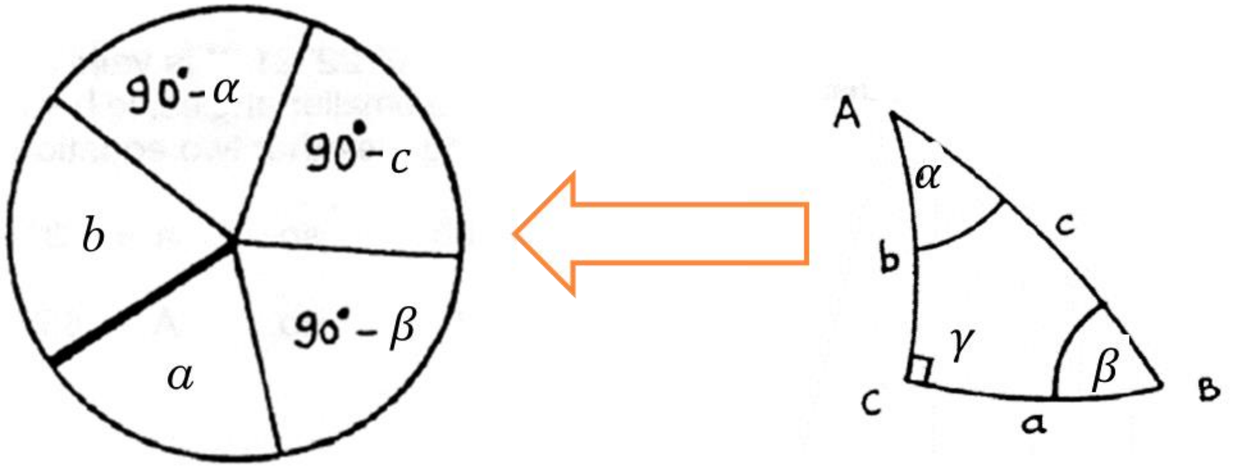


Solution of Right-Angle Spherical Triangle

- **Napier's Rule**

$$\sin(90-\beta) = \tan(90-c) \times \tan(\alpha)$$

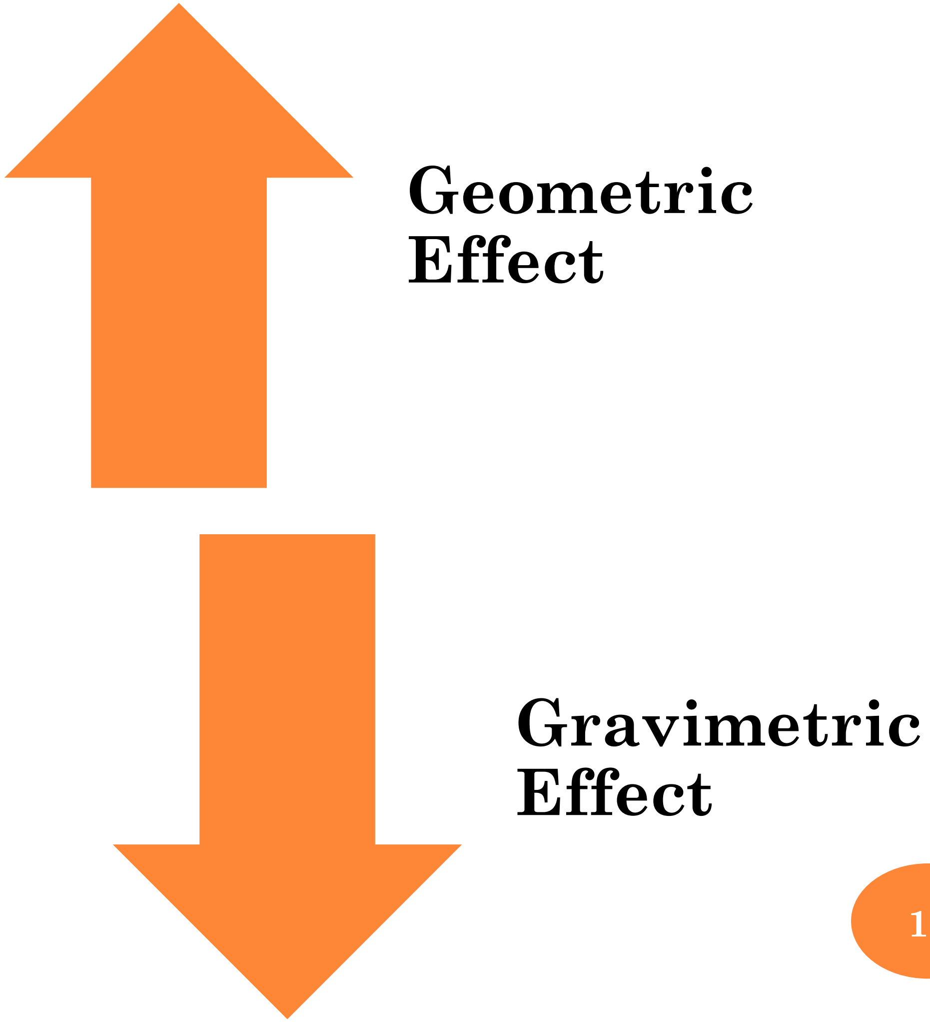
$$\sin(90-\alpha) = \cos a \times \cos(90 - \beta)$$



Spherical Excess

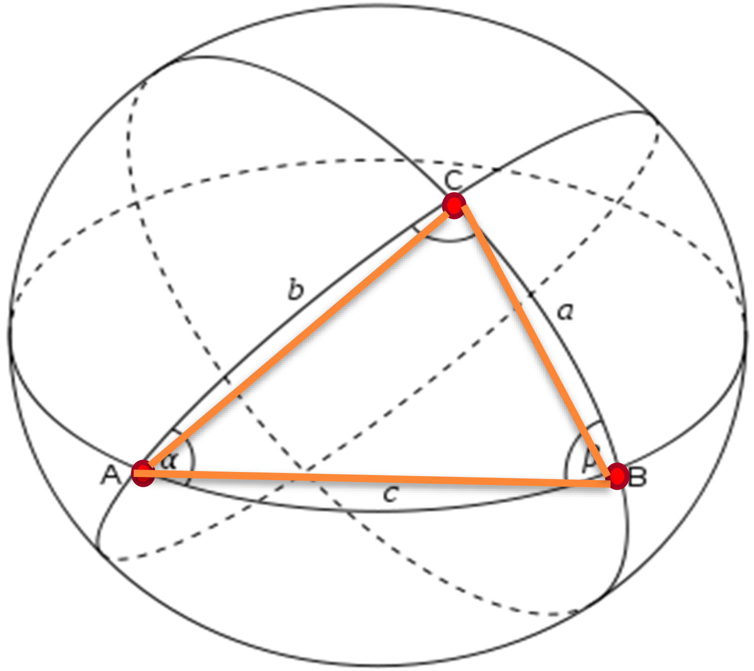
Effects of Earth Representation on Observations

- **Geometric Effect**
  - Any correction related to the earth curvature.
  - It includes: -
    1. Spherical excess
    2. Convergence of meridians
- **Gravimetric Effect** (Reduction to Ellipsoid)
  - Corrections related to the reduction of observations from earth's surface to ellipsoid.
  - It mainly depends upon the deflection of the vertical components.



SPHERICAL EXCESS

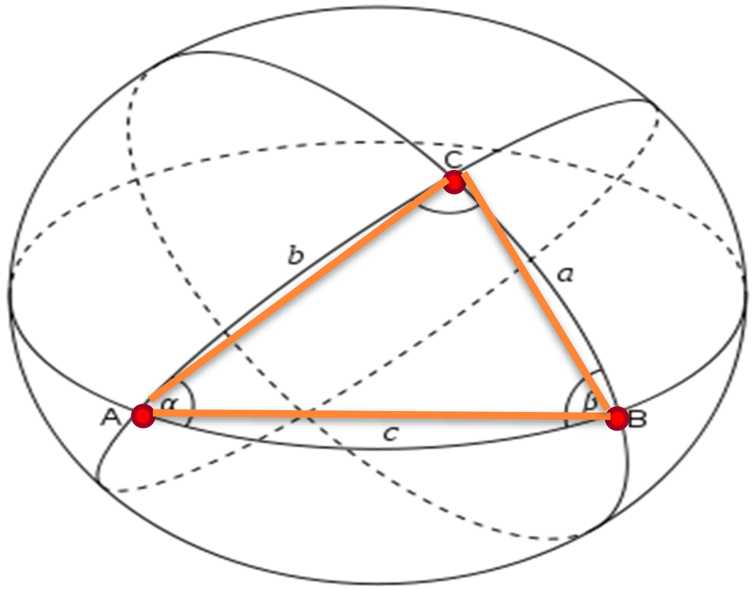
- On the plane, the sum of the interior angles of any triangle is exactly 180°.
- On a sphere, the corresponding sum is always greater than 180°. That is, $180° < \alpha + \beta + \gamma$
- The positive quantity $\varepsilon = \alpha + \beta + \gamma - 180°$ is called the **spherical excess** of the triangle.



Spherical Excess

- **Auxiliary plane triangle**

The Legendre's theory states that the spherical triangle is equivalent to a plane triangle where the lengths of sides in both triangles are equals while the angles differ from each other by 1/3 of the spherical excess as follows: -



$A = \alpha - \frac{\varepsilon}{3}$, $B = \beta - \frac{\varepsilon}{3}$, and $C = \gamma - \frac{\varepsilon}{3}$.

Such that ε is computed as a function of the triangle's sides' lengths.

SPHERICAL EXCESS - COMPUTATION

- **Equation of Spherical Excess**

$\varepsilon'' = \frac{F}{R^2 \sin 1''}$, ε *in seconds.*

F: Square **area** of the spherical triangle in km^2.

R: Mean earth's radius in km ≈ 6370 km.

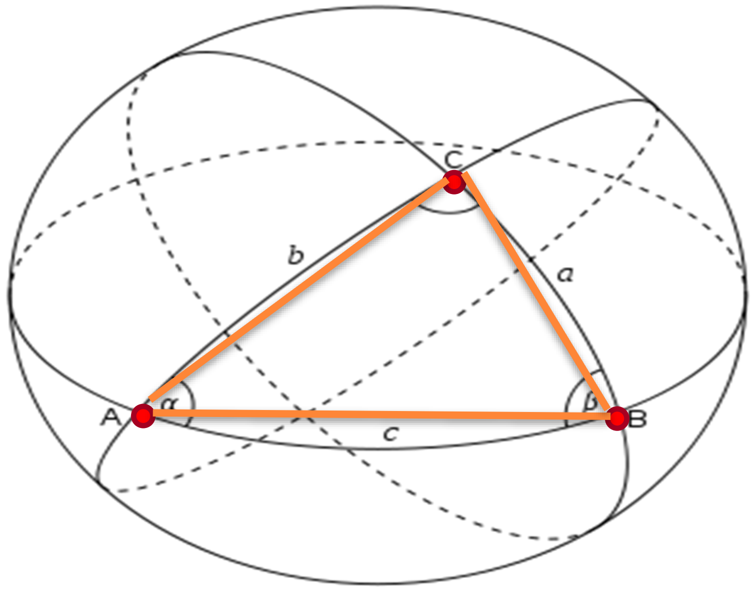
Another equation is: -

$$\tan\frac{1}{4}\varepsilon = \sqrt{\tan\frac{s}{2}\tan\frac{s-a}{2}\tan\frac{s-b}{2}\tan\frac{s-c}{2}}$$

Where,

$s = \frac{a+b+c}{2}$,

a, b, and c: The sides of the spherical triangle.



Spherical Excess - Computation

- **Cases of Spherical Excess**

1. If the lengths of the sides of the triangle are less than 10 km or the square area is less than 25000 km^2, spherical excess is neglected.
2. When the side lengths of triangle are more than 160 km, ellipsoidal excess ε_1 is considered such that: -

$$\varepsilon_1 = \varepsilon'' \left[1 + \frac{a^2 + b^2 + c^2}{24\,MN}\right],\ \varepsilon\ in\ seconds.$$

ε'': *The spherical excess.*

M: Radius of curvature in meridian direction.

N: Radius of curvature in prime vertical.

a, b, and c: sides of triangle.

Spherical Excess – Numerical Exercise

From two known stations A and B, a third station C to the east of them was fixed by observation of all the angles. If the length of side AB = 28866.149 meters, mean radius of curvature in the area = 6369750 meters, and the observed angles are: -

A = 59° 34' 16.14"

B = 52° 13' 22.00"

C = 68° 12' 25 .04"

Calculate the Spherical Excess and the triangular misclosure.

Spherical Excess – Numerical Exercise

First, the area of triangle is $\frac{1}{2}c^2 . \frac{\sin\alpha \times \sin\beta}{\sin(\alpha+\beta)}$

Therefore, spherical excess is: $\varepsilon'' = \frac{F}{R^2 \sin 1''}$, ε *in seconds.*

$$\varepsilon'' = 1.55\ seconds$$

The internal angles corrected for spherical excess are: -

$A = \alpha - \frac{\varepsilon}{3}$, $B = \beta - \frac{\varepsilon}{3}$, and $C = \gamma - \frac{\varepsilon}{3}$

The closing error (c.e) is given by:

$c.e = \sum(angles) - \varepsilon - 180^{o} = 1.69''$

The internal angles corrected for closing error are: -

$\bar{A} = A - \frac{c.e}{3}$,$\bar{B} = B - \frac{c.e}{3}$, and $\bar{C} = C - \frac{c.e}{3}$ (be careful with the sign of c.e correction)

Real-World Applications of Spherical Triangles

- Spherical triangles and related concepts has several applications, particularly in fields such as:

a) **Geodesy**

- Calculation of distances, angles, and positions on the Earth's surface.
- Satellite Navigation Systems: Spherical triangles are used in the trilateration process, where signals from multiple satellites are intersected to calculate the receiver's position.

b) **Celestial Navigation**

- Sailors and aviators use celestial navigation to determine their position using celestial bodies such as the sun, moon, stars, and planets.

c) **Map Projections and Cartography**

- Spherical triangles play a crucial role in understanding and developing map projection techniques, as they provide the foundation for preserving angles, distances, and shapes on maps.

d) **Astronomy**

- Astronomers employ spherical triangles to calculate positions, parallaxes, and angular separations between stars and other celestial bodies.

Can you Compare Between Plane and Geodetic Surveying?

Plane and Geodetic Triangles

Parameter	Plane Surveying	Geodetic Surveying
Shape of Earth	Plane	Sphere, Spheroid, or Ellipsoid
Distances	Straight Lines ($\leq$ 10 km)	curved lines (> 10 km)
Triangles	Plane triangles	Spherical or ellipsoidal triangles
Coverage	Small-scale projects	Large-scale projects
Desired accuracy	Low	High

- **Can you generalize this comparison?**
- **Key differences may include:**
  - Point positions,
  - Lengths(distance in either shape or magnitude)
  - Azimuth definition,
  - Coverage,
  - Earth representation,
  - Scope
  - ……

LET'S SUMMARIZE

Quiz (1)

Lecture 1 to Lecture 3

To join, Please open "Quizzes" app on your Smart Phone and type in the PIN

THANK YOU

End of Presentation